

THE j -MULTIPLICITY OF MONOMIAL IDEALS

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ABSTRACT. We prove a characterization of the j -multiplicity of a monomial ideal as the normalized volume of a polytopal complex. Our result is an extension of Tessier's volume-theoretic interpretation of the Hilbert-Samuel multiplicity for \mathfrak{m} -primary monomial ideals. We also give a description of the ε -multiplicity of a monomial ideal in terms of the volume of a region.

1. INTRODUCTION

The j -multiplicity was defined in 1993 by Achilles and Manaresi in [1] as a generalization of the Hilbert-Samuel multiplicity for arbitrary ideals in a Noetherian local ring. Several results on the Hilbert-Samuel multiplicity have been successfully extended to more general classes of ideals using the j -multiplicity, for example [8], [14], and [4]. The main result of this paper may be viewed as one of these extensions.

Let (R, \mathfrak{m}, k) be a Noetherian local ring of dimension d , and $I \subset R$ an ideal. The j -multiplicity of I is defined as the limit

$$j(I) = \lim_{n \rightarrow \infty} \frac{(d-1)!}{n^{d-1}} \lambda_R(\mathrm{H}_{\mathfrak{m}}^0(I^n/I^{n+1})).$$

There have been previous approaches for computing the j -multiplicity. For example, in [1] and [21] it is proven that if k is infinite, then for general elements a_1, \dots, a_d in I , and $\alpha = (a_1, \dots, a_{d-1})$, we have

$$j(I) = \lambda_R(R/((\alpha :_R I^\infty) + a_d R)).$$

This formula is applied to compute specific examples in [12].

Let R denote now the polynomial ring $k[x_1, \dots, x_d]$ over the field k , \mathfrak{m} the homogeneous maximal ideal (x_1, \dots, x_d) , and I a monomial ideal of R . The *Newton polyhedron* of I is the convex hull of the points in \mathbb{R}^d that correspond to monomials in I , which we will denote by $\mathrm{conv}(I)$. In this paper we generalize the classical result that describes the Hilbert-Samuel multiplicity of an \mathfrak{m} -primary ideal as the normalized volume of the complement of its Newton polyhedron in $\mathbb{R}_{\geq 0}^d$, see [17]. If I is not \mathfrak{m} -primary, the complement of $\mathrm{conv}(I)$ is infinite, but we can define the analogue of this region in the general case by considering the truncated cone from the origin to the union of the bounded faces of $\mathrm{conv}(I)$. This truncated cone will be denoted by $\mathrm{pyr}(I)$. With this notation, we can state our main result:

Theorem 3.2. *Let $I \subset R$ be a monomial ideal. Then $j(I) = d! \mathrm{vol}(\mathrm{pyr}(I))$.*

Earlier unpublished work of J. Validashti obtains this formula in dimension two. The rest of the paper is organized as follows: In the second section we set up the notation and also present some results that will be used in the proof of the main theorem. The third section will include the proof of Theorem 3.2. In the

fourth section we provide an extension of this result to pointed normal affine toric varieties. In the fifth section we will apply our characterization of the saturation of a monomial ideal in R in Lemma 2.2 to give a geometric description of the ε -multiplicity. The paper ends with some examples in a sixth section.

2. PRELIMINARIES

Let $R = k[x_1, \dots, x_d]$ be a polynomial ring over a field k and $\mathfrak{m} = (x_1, \dots, x_d)$ its homogeneous maximal ideal. Let I be a monomial ideal of R minimally generated by $x^{v_1}, x^{v_2}, \dots, x^{v_n}$ where $v_i = (v_{i,1}, \dots, v_{i,d})$ and $x^{v_i} = x_1^{v_{i,1}} \cdots x_d^{v_{i,d}}$. For a monomial ideal L in R we denote by $\Gamma(L)$ the set of lattice points in \mathbb{R}^d corresponding to the exponents. Additionally, if $L_1 \supseteq L_2$ are monomial ideals, we will write $\Gamma(L_1/L_2)$ for $\Gamma(L_1) \setminus \Gamma(L_2)$.

We denote by $\text{conv}(I)$ the *Newton polyhedron* of I , that is:

$$\text{conv}(I) := \text{conv}(v_1, \dots, v_n) + \mathbb{R}_{\geq 0}^d,$$

where $+$ denotes the Minkowski sum. It is worth noting that the collection of bounded facets of the Newton polyhedron is not convex, and thus is not a polytope, but rather has the structure of a polytopal complex. Notice also that $\text{conv}(I) = \text{conv}(\Gamma(I))$. Since every polyhedron is defined by the intersection of finitely many closed half spaces, we can define $H_i = \{x \in \mathbb{R}^n \mid \langle x, b_i \rangle = c_i\}$, with $b_i \in \mathbb{Q}^d$, $c_i \in \mathbb{Q}$ for $i = 1, \dots, w$ to be the *supporting hyperplanes* of $\text{conv}(I)$ such that

$$\text{conv}(I) = H_1^+ \cap H_2^+ \cap \cdots \cap H_w^+,$$

where $H_i^+ = \{x \in \mathbb{R}^n \mid \langle x, b_i \rangle \geq c_i\}$. Let $\mathcal{F}_i = H_i \cap \text{conv}(I)$ for $i = 1, \dots, h$ be the facets of $\text{conv}(I)$. We will assume that H_1, \dots, H_u , are the hyperplanes corresponding to unbounded facets.

It can be shown that all the vector b_i have nonnegative components, and that $b_i \in \mathbb{R}_{\geq 0}^d$ if and only if \mathcal{F}_i is a bounded facet, as in [15, Lemma 1.1]. This forces the c_i to be nonnegative, and in fact positive in the case of a bounded facet.

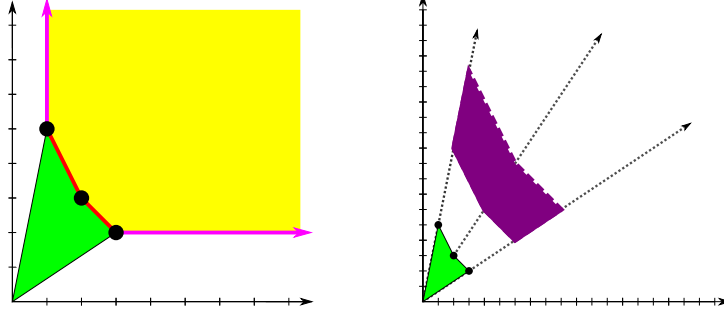
We denote by $\text{vert}(I)$ the set of vertices of $\text{conv}(I)$, and set $\text{bd}(I) = \bigcup_{i=u+1}^w \mathcal{F}_i$ for the union of the bounded facets of $\text{conv}(I)$. If \mathcal{P} is a polytope, we will write $\text{pyr}(\mathcal{P})$ for $\text{conv}(\mathcal{P}, \mathbf{0})$, the truncated cone, or pyramid, over \mathcal{P} . By abuse of notation, we will write $\text{pyr}(I)$ for $\bigcup_{i=u+1}^w \text{pyr}(\mathcal{F}_i)$. Note that the monomials corresponding to the points in $\text{vert}(I)$ are part of the set of minimal generators of I , so we will assume $\text{vert}(I) = \{v_1, \dots, v_s\}$ for some $1 \leq s \leq n$. We will also find it convenient to define the n^{th} *cone section* of a polytope \mathcal{P} as

$$\text{cone}_n(\mathcal{P}) := ((n+1)\text{pyr}(\mathcal{P}) \setminus (n+1)\mathcal{P}) \setminus (n\text{pyr}(\mathcal{P}) \setminus n\mathcal{P}),$$

which we may alternatively write as $\bigcup_{n \leq s \leq n+1} s\mathcal{P}$. We again abuse notation by writing $\text{cone}_n(I)$ for $\bigcup_{i=u+1}^w \text{cone}_n(\mathcal{F}_i)$.

For the monomial ideal $I = (xy^5, x^2y^3, x^3y^2)$, in Figure 1, on the left we mark $\text{vert}(I)$ with black dots, $\text{bd}(I)$ with dark red lines, and the unbounded facets with pink lines. In this example, the green region with its boundary forms $\text{pyr}(I)$, and the yellow region with its boundary forms $\text{conv}(I)$. In the graph on the right, we shade $\text{cone}_2(I)$ in purple, where the top dotted segments of the boundary are not included.

The following description of the faces of the Newton polyhedron will be useful.

FIGURE 1. Various regions for the monomial ideal $I = (xy^5, x^2y^3, x^3y^2)$

Lemma 2.1. [15, Lemma 3.1] *Let \mathcal{F} be a face of $\text{conv}(I)$ with supporting hyperplane $H = \{x \in \mathbb{R}^n \mid \langle x, b \rangle = c\}$. Then $\mathcal{F} \cap \text{vert}(I) = \{v_{i_1}, \dots, v_{i_r}\}$ is non-empty, and*

$$\mathcal{F} = \text{conv}(v_{i_1}, \dots, v_{i_r}) + \sum_{j: b_j = 0} \mathbb{R}_{\geq 0} e_j$$

where e_j is the unit vector with nonzero j^{th} component.

Recall that for any submodule N of an R -module M , the *saturation* of N , denoted $(N :_M \mathfrak{m}^\infty)$, is the set of elements a in M for which there exists $n \in \mathbb{N}$ such that $am^n \in N$. The *zeroth local cohomology module* of M is defined to be $(0 :_M \mathfrak{m}^\infty)$ and is denoted by $H_{\mathfrak{m}}^0(M)$. Notice that in general, $H_{\mathfrak{m}}^0(M/N) = (N :_M \mathfrak{m}^\infty)/N$. If I is a monomial ideal, then $(I :_R \mathfrak{m}^\infty)$ is also monomial.

The *integral closure* of an arbitrary ideal J is the set of elements x in R that satisfy an integral relation $x^n + a_1x^{n-1} + \dots + a_{n-1}x + a_n$ where $a_i \in J^i$ for $i = 1, \dots, n$. It is denoted by \bar{J} and it is an ideal. For monomial ideals, it is possible to give a geometric description of the integral closure, namely $\Gamma(\bar{I}) = \mathbb{Z}^d \cap \text{conv}(I)$, i.e., $\text{conv}(\bar{I}) = \text{conv}(I)$; see [19, Proposition 7.25].

Proposition 2.2. $\Gamma(\bar{I} :_R \mathfrak{m}^\infty) = H_1^+ \cap \dots \cap H_u^+ \cap \mathbb{Z}_{\geq 0}^d$.

Proof. Let $v \in H_1^+ \cap \dots \cap H_u^+$; then $\langle v, b_i \rangle \geq c_i$ for $i = 1, \dots, u$. For $t \in \mathbb{R}_{\geq 0}$, one has

$$\langle v + te_j, b_i \rangle \geq c_i + tb_{i,j} \geq c_i$$

for $j = 1, \dots, d$ and $i = 1, \dots, u$. Also, if $u+1 \leq i \leq w$ then all the entries of b_i are positive, so we also have that $\langle v + te_j, b_i \rangle \geq c_i$ for $t \gg 0$. We conclude $x_j^t x^v \in \bar{I}$ for $j = 1, \dots, d$ and $t \gg 0$, that is $v \in \Gamma(\bar{I} :_R \mathfrak{m}^\infty)$.

Conversely, if $v \in \Gamma(\bar{I} :_R \mathfrak{m}^\infty)$, then $v + te_j \in \text{conv}(I) \subset H_1^+ \cap \dots \cap H_u^+$ for $j = 1, \dots, d$ and $t \gg 0$. Now, suppose $v \notin H_i^+$ for some $1 \leq i \leq u$, then $\langle v, b_i \rangle < c_i$. By Lemma 2.1, since \mathcal{F}_i is an unbounded facet, we can pick j such that $b_{i,j} = 0$, and hence $\langle v + te_j, b_i \rangle = \langle v, b_i \rangle < c_i$ for every $t \in \mathbb{R}$, which is a contradiction. \square

Recall that the *analytic spread* of an ideal I , denoted by $l(I)$, is defined to be the dimension of its *special fiber ring* $\mathcal{F}(I) = \text{gr}_I(R) \otimes_R R/\mathfrak{m} = \bigoplus_{n=0}^{\infty} I^n/\mathfrak{m}I^n$, where $\text{gr}_I(R)$ is the associated graded algebra of I , i.e., $\text{gr}_I(R) = \bigoplus_{n=0}^{\infty} I^n/I^{n+1}$. We will say that I has *maximal analytic spread* if $l(I) = \dim(R)$.

Lemma 2.3. $\Gamma(\bar{I} :_R \mathfrak{m}^{\infty}) \subset \text{conv}(I) \cup \text{pyr}(I)$.

Proof. The result follows immediately by Proposition 2.2 if $\text{conv}(I)$ does not have bounded facets. We will assume that $u < w$.

Let v be a nonzero vector in $\mathbb{Z}_{\geq 0}^d \setminus (\text{pyr}(I) \cup \text{conv}(I))$. We will proceed by contradiction. Suppose $v \in \Gamma(\bar{I} :_R \mathfrak{m}^{\infty})$, then $v \in H_1^+ \cap \dots \cap H_u^+$ by Proposition 2.2.

Note that since b_i has positive entries for each $i = u+1, \dots, w$, we have $\langle v, b_i \rangle \neq 0$. For each $u+1 \leq i \leq w$ we can find a real number t_i such that $t_i v \in H_i$, each of which is positive because $c_i > 0$. Suppose, without loss of generality, that $t_w = \max\{t_{u+1}, \dots, t_w\}$. Since $v \notin H_i^+$ for some $u+1 \leq i \leq w$ we have $\langle v, b_i \rangle < c_i$, then $t_i > 1$, and so $t_w > 1$.

Now, $\langle t_w v, b_i \rangle \geq \langle t_i v, b_i \rangle = c_i$ for $i = u+1, \dots, w$, so

$$t_w v \in H_1^+ \cap H_2^+ \cap \dots \cap H_{w-1}^+ \cap H_w = F_w.$$

Then we have $v \in \text{pyr}(I)$ which is a contradiction. \square

Remark 2.4. By [15, Corollary 3.4], $n \text{conv}(I) = \text{conv}(I^n)$, $n \text{pyr}(I) = \text{pyr}(I^n)$, and $n \text{bd}(I) = \text{bd}(I^n)$ for every $n \geq 1$. It follows from this and the previous lemma that $\Gamma((\bar{I}^{n+1} :_{\bar{I}^n} \mathfrak{m}^{\infty})/\bar{I}^{n+1}) \subseteq \text{cone}_n(I)$.

In the following lemma, we will use the notion of the *Hausdorff distance* between compact sets A and B in \mathbb{R}^d , which is defined as

$$\rho(A, B) := \inf\{\lambda \geq 0 \mid A \subseteq B + \lambda U, B \subseteq A + \lambda U\},$$

where U is the unit ball. We will use a related notion for polytopes: for a convex polytope $\mathcal{P} = \text{conv}(v_1, \dots, v_t)$ in \mathbb{R}^d , we will say that another convex polytope with t vertices $\mathcal{P}' = \text{conv}(v'_1, \dots, v'_t)$ is an ε -*shaking* of \mathcal{P} if $|v_j - v'_j| < \varepsilon$ for all j .

Lemma 2.5. Fix $\mathcal{P} = \text{conv}(v_1, \dots, v_t)$ in \mathbb{R}^d . Let $(\mathcal{P}_1^{(n)}, \dots, \mathcal{P}_s^{(n)})_{n \in \mathbb{N}}$ be a sequence of s -tuples of polytopes such that each $\mathcal{P}_j^{(n)}$ is a $(1/n)$ -shaking of \mathcal{P} . Then,

$$\lim_{n \rightarrow \infty} \text{vol} \left(\mathcal{P} \cap \bigcap_{i=1}^s \mathcal{P}_i^{(n)} \right) = \text{vol}(\mathcal{P}).$$

Proof. Let $\mathcal{P}' = \text{conv}(v'_1, \dots, v'_t)$ be an ε -shaking of \mathcal{P} , and write

$$\mathcal{P}' \amalg \mathcal{P} = \text{conv}(v'_1, \dots, v'_t, v_1, \dots, v_t).$$

Note that $\mathcal{P}' \cup \mathcal{P} \subseteq \mathcal{P}' \amalg \mathcal{P}$. Also, $\rho(\mathcal{P}, \mathcal{P}' \amalg \mathcal{P}) < \varepsilon$, since for $q \in \mathcal{P}' \amalg \mathcal{P}$, we may write $q = \sum \lambda_j v_j + \sum \lambda'_j v'_j$ with $\sum \lambda_i + \sum \lambda'_i = 1$, and

$$|q - \sum \lambda_j v_j - \sum \lambda'_j v'_j| = |\sum \lambda'_j v'_j - \sum \lambda'_j v_j| \leq \sum \lambda'_j |v'_j - v_j| < \varepsilon,$$

where $\sum \lambda_j v_j + \sum \lambda'_j v'_j \in \mathcal{P}$. Similarly $\rho(\mathcal{P}, \mathcal{P}') < \varepsilon$.

We have

$$\begin{aligned}
0 \leq \text{vol}(\mathcal{P}) - \text{vol}(\mathcal{P} \cap \bigcap_{i=1}^s \mathcal{P}_i^{(n)}) &\leq \sum_{i=1}^s (\text{vol}(\mathcal{P}) - \text{vol}(\mathcal{P} \cap \mathcal{P}_i^{(n)})) \\
&= \sum_{i=1}^s (\text{vol}(\mathcal{P} \cup \mathcal{P}_i^{(n)}) - \text{vol}(\mathcal{P}_i^{(n)})) \\
&\leq \sum_{i=1}^s (\text{vol}(\mathcal{P} \amalg \mathcal{P}_i^{(n)}) - \text{vol}(\mathcal{P}_i^{(n)})).
\end{aligned}$$

Then, by continuity of volume with respect to Hausdorff distance, see [20, Theorem 6.2.17], we have that $\text{vol}(\mathcal{P}_i^{(n)})$ and $\text{vol}(\mathcal{P} \amalg \mathcal{P}_i^{(n)})$ both converge to $\text{vol}(\mathcal{P})$ as $n \rightarrow \infty$. Thus, $\text{vol}(\mathcal{P} \cap \bigcap_{i=1}^s \mathcal{P}_i^{(n)}) \rightarrow \text{vol}(\mathcal{P})$ as $n \rightarrow \infty$, as required. \square

Recall that the *Ehrhart function* of a polytope $\mathcal{P} \subset \mathbb{R}^d$ is defined as

$$E_{\mathcal{P}}(n) := \#(\mathbb{Z}^d \cap n\mathcal{P}).$$

Ehrhart [7] showed that if the vertices of \mathcal{P} have integer coordinates, $E_{\mathcal{P}}(n)$ is a polynomial of degree $\dim(\mathcal{P})$ with leading coefficient equal to the relative volume of \mathcal{P} (cf., [11, Chapter 12]). Recall that a function $f: \mathbb{N} \rightarrow \mathbb{Z}$ is called a *quasi-polynomial* if there is an $m \in \mathbb{N}$ and polynomials f_0, \dots, f_{m-1} such that $f(n) = f_{(n \bmod m)}(n)$ for all $n \in \mathbb{N}$. If \mathcal{P} has vertices in \mathbb{Q}^d , then the Ehrhart function $E_{\mathcal{P}}$ is a quasi-polynomial. We will employ a strengthening of this fact.

Proposition 2.6. *Let $\mathcal{P} \subset \mathbb{R}^d$ be a $(d-1)$ -dimensional polytope with vertices in \mathbb{Q}^d . Suppose that the affine span of \mathcal{P} contains a point in the integer lattice \mathbb{Z}^d . Then $\#(\mathbb{Z}^d \cap \text{cone}_n(\mathcal{P}))$ as a function of n is given by a quasi-polynomial of the form*

$$p(n) = d \text{vol}(\text{pyr}(\mathcal{P}))n^{d-1} + O(n^{d-2}).$$

Proof. Write

$$\begin{aligned}
\#(\mathbb{Z}^d \cap \text{cone}_n(\mathcal{P})) &= \# \left(\mathbb{Z}^d \cap \left(((n+1)\text{pyr}(\mathcal{P}) \setminus (n+1)\mathcal{P}) \setminus (n\text{pyr}(\mathcal{P}) \setminus n\mathcal{P}) \right) \right) \\
&= (E_{\text{pyr}(\mathcal{P})}(n+1) - E_{\text{pyr}(\mathcal{P})}(n)) - (E_{\mathcal{P}}(n+1) - E_{\mathcal{P}}(n)).
\end{aligned}$$

By hypothesis, the affine span of each $(d-1)$ -dimensional face of $\text{pyr}(\mathcal{P})$ contains a lattice point, because the affine span of \mathcal{P} has a lattice point and every other $(d-1)$ -dimensional face contains $\mathbf{0}$. A conjecture of Ehrhart [7], proved by McMullen [10] and Stanley [16] (see also [3] for a proof based on monomial ideal techniques), confirms that in this situation $E_{\text{pyr}(\mathcal{P})}$ and $E_{\mathcal{P}}$ are quasi-polynomials of the form

$$\begin{aligned}
E_{\text{pyr}(\mathcal{P})}(n) &= \text{vol}(\text{pyr}(\mathcal{P}))n^d + a_{d-1}n^{d-1} + O(n^{d-2}) \\
E_{\mathcal{P}}(n) &= b_{d-1}n^{d-1} + O(n^{d-2}),
\end{aligned}$$

where a_{d-1} and b_{d-1} are constants; specifically, they do not depend on n . The claim follows from the formula above. \square

Remark 2.7. We can apply the result of the previous proposition to each bounded facet \mathcal{F}_i of $\text{conv}(I)$. Using the definition $\text{cone}_n(I) = \bigcup_{u+1}^h \text{cone}_n(\mathcal{F}_i)$, we can conclude that $\#(\mathbb{Z}^d \cap \text{cone}_n(I))$ is given by a polynomial of the form

$$p(n) = d \text{vol}(\text{pyr}(I))n^{d-1} + O(n^{d-2}),$$

as in the proposition. Notice that double-counting of the points in the face in common of two regions $\text{cone}_n(\mathcal{F}_i)$ for distinct i does not change the leading coefficient of the polynomial, because these faces have lower dimension and the degree of the Ehrhart polynomial of a polytope is equal to its dimension.

3. THE j -MULTIPLICITY OF MONOMIAL IDEALS

In order to be consistent with the definition of j -multiplicity, which is defined for ideals in a local ring, in this chapter we will consider $R = k[x_1, \dots, x_d]_{\mathfrak{m}}$ and I an ideal generated by monomials. All the results of the second section still hold in this setting, because all the ideals involved are monomial ideals. Moreover, the analytic spread does not change.

For an R -module M , we can define the j -multiplicity of I with respect to M as

$$j(I, M) = \lim_{n \rightarrow \infty} \frac{(d-1)!}{n^{d-1}} \lambda_R(\mathrm{H}_{\mathfrak{m}}^0(I^n M / I^{n+1} M)).$$

In the case $M = R$, $j(I, R)$ will be denoted $j(I)$ as in the introduction.

The following proposition shows that we can compute $j(I)$ using the filtration $\{\overline{I^n}\}_{n \in \mathbb{N}}$. The proof is similar to [8, Proposition 2.10].

Proposition 3.1. *Let I be a monomial ideal, then*

$$j(I) = \lim_{n \rightarrow \infty} \frac{(d-1)!}{n^{d-1}} \lambda_R(\mathrm{H}_{\mathfrak{m}}^0(\overline{I^n} / \overline{I^{n+1}})).$$

Proof. By [19, Theorem 7.29], $\overline{I^{n+1}} = \overline{I^n}$ for $n \geq d$. From [12, Theorem 3.11], and the following exact sequence of R -modules

$$0 \rightarrow \overline{I^d} \rightarrow R \rightarrow R/\overline{I^d} \rightarrow 0,$$

we obtain $j(I, R) = j(I, \overline{I^d}) + j(I, R/\overline{I^d})$. But $I^d(R/\overline{I^d}) = 0$, so $j(I, R/\overline{I^d}) = 0$ and then $j(I) = j(I, R) = j(I, \overline{I^d})$.

Now,

$$\begin{aligned} j(I) = j(I, \overline{I^d}) &= \lim_{t \rightarrow \infty} \frac{(d-1)!}{t^{d-1}} \lambda_R(\mathrm{H}_{\mathfrak{m}}^0(I^t \overline{I^d} / I^{t+1} \overline{I^d})) \\ &= \lim_{t \rightarrow \infty} \frac{(d-1)!}{t^{d-1}} \lambda_R(\mathrm{H}_{\mathfrak{m}}^0(\overline{I^{t+d}} / \overline{I^{t+d+1}})) \\ &= \lim_{n \rightarrow \infty} \frac{n^{d-1}}{(n-d)^{d-1}} \frac{(d-1)!}{n^{d-1}} \lambda_R(\mathrm{H}_{\mathfrak{m}}^0(\overline{I^n} / \overline{I^{n+1}})), \end{aligned}$$

and the result follows from the equality $\lim_{n \rightarrow \infty} \frac{n^{d-1}}{(n-d)^{d-1}} = 1$. \square

Theorem 3.2. *Let $I \subset R$ be a monomial ideal. Then $j(I) = d! \text{vol}(\text{pyr}(I))$.*

Proof. By Lemma 3.1, we compute

$$j(I) = \lim_{n \rightarrow \infty} \frac{(d-1)!}{n^{d-1}} \lambda_R(\mathrm{H}_{\mathfrak{m}}^0(\overline{I^n} / \overline{I^{n+1}})).$$

Step 1: The proof that $j(I) \leq d! \text{vol}(\text{pyr}(I))$:

Recall that $H_{\mathfrak{m}}^0(\overline{I^n}/\overline{I^{n+1}}) = ((\overline{I^{n+1}} : \mathfrak{m}^\infty) \cap \overline{I^n})/\overline{I^{n+1}}$, so that

$$\begin{aligned} \lambda_R(H_{\mathfrak{m}}^0(\overline{I^n}/\overline{I^{n+1}})) &= \#(\mathbb{Z}^d \cap (\Gamma(\overline{I^{n+1}} : \mathfrak{m}^\infty) \cap \text{conv}(I^n) \setminus \text{conv}(I^{n+1}))) \\ &\leq \#(\mathbb{Z}^d \cap (\text{conv}(I^{n+1}) \cup \text{pyr}(I^{n+1})) \cap \text{conv}(I^n) \setminus \text{conv}(I^{n+1})) \end{aligned}$$

where the last inequality holds by Lemma 2.3. By [15, Corollary 3.4], we have $n \text{conv}(I) = \text{conv}(I^n)$, $n \text{pyr}(I) = \text{pyr}(I^n)$, and $n \text{bd}(I) = \text{bd}(I^n)$ for every $n \geq 1$. Note that $\text{bd}(I) = \text{conv}(I) \cap \text{pyr}(I)$; then,

$$\begin{aligned} &(\text{pyr}(I^{n+1}) \cap \text{conv}(I^n)) \setminus \text{conv}(I^{n+1}) \\ &= ((n+1) \text{pyr}(I) \cap n \text{conv}(I)) \setminus (n+1) \text{conv}(I) \\ &= ((n+1) \text{pyr}(I) \cap n \text{conv}(I)) \setminus (n+1) \text{bd}(I) \\ &= ((n+1) \text{pyr}(I) \setminus (n+1) \text{bd}(I)) \setminus (n \text{pyr}(I) \setminus n \text{bd}(I)) = \text{cone}_n(I). \end{aligned}$$

It follows that $\lambda_R(H_{\mathfrak{m}}^0(\overline{I^n}/\overline{I^{n+1}})) \leq \#(\mathbb{Z}^d \cap \text{cone}_n(I))$. Therefore, by Remark 2.7 ,

$$j(I) = \lim_{n \rightarrow \infty} \frac{(d-1)!}{n^{d-1}} \lambda_R(H_{\mathfrak{m}}^0(\overline{I^n}/\overline{I^{n+1}})) \leq d! \text{vol}(\text{pyr}(I)).$$

Step 2: The proof that $j(I) \geq d! \text{vol}(\text{pyr}(I))$:

Step 2a: Reduction to the case of an ideal corresponding to a single facet:

First we claim that it suffices to verify the inequality for a monomial ideal whose Newton polyhedron has a single bounded facet. Indeed, if the inequality holds for such ideals, write $J_1, \dots, J_t \subset I$ for the monomial ideals corresponding to the bounded facets of I and $\mathcal{P}_1, \dots, \mathcal{P}_t$ for the corresponding facets, so that we have $\text{bd}(I) = \bigcup_i \mathcal{P}_i$ and $\text{bd}(J_i) = \mathcal{P}_i$. Then since we have

$$\bigcup_{i=1}^t \Gamma((\overline{J_i^{n+1}} : \overline{J_i^n} \mathfrak{m}^\infty)/\overline{J_i^{n+1}}) \subseteq \Gamma((\overline{I^{n+1}} : \overline{I^n} \mathfrak{m}^\infty)/\overline{I^{n+1}}),$$

and

$$\bigcup_{i=1}^t \text{cone}_n(\mathcal{P}_i) = \text{cone}_n(I),$$

we have a containment

$$\begin{aligned} \text{cone}_n(I) \setminus \Gamma((\overline{I^{n+1}} : \overline{I^n} \mathfrak{m}^\infty)/\overline{I^{n+1}}) &\subseteq \bigcup_{i=1}^t \text{cone}_n(\mathcal{P}_i) \setminus \bigcup_{i=1}^t \Gamma((\overline{J_i^{n+1}} : \overline{J_i^n} \mathfrak{m}^\infty)/\overline{J_i^{n+1}}) \\ &\subseteq \bigcup_{i=1}^t (\text{pyr } n(\mathcal{P}_i) \setminus \Gamma((\overline{J_i^{n+1}} : \overline{J_i^n} \mathfrak{m}^\infty)/\overline{J_i^{n+1}})). \end{aligned}$$

Notice $\#\Gamma((\overline{J^{n+1}} : \overline{J^n} \mathfrak{m}^\infty)/\overline{J^{n+1}}) = \lambda_R(H_{\mathfrak{m}}^0(\overline{J^n}/\overline{J^{n+1}}))$ for any ideal J , so

$$\begin{aligned} &\#(\text{cone}_n(I) \cap \mathbb{Z}^d) - \lambda_R(H_{\mathfrak{m}}^0(\overline{I^n}/\overline{I^{n+1}})) \\ &\leq \sum_{i=1}^t \left(\#(\text{cone}_n(\mathcal{P}_i) \cap \mathbb{Z}^d) - \lambda_R(H_{\mathfrak{m}}^0(\overline{J_i^n}/\overline{J_i^{n+1}})) \right). \end{aligned}$$

Thus, if the claimed inequality holds for each J_i , we have that

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{(d-1)!}{n^{d-1}} \left(\#(\text{cone}_n(I) \cap \mathbb{Z}^d) - \lambda_R(\mathbf{H}_m^0(\overline{I^n}/\overline{I^{n+1}})) \right) \\ & \leq \sum_i \left(\lim_{n \rightarrow \infty} \frac{(d-1)!}{n^{d-1}} \#(\text{cone}_n(\mathcal{P}_i) \cap \mathbb{Z}^d) - \lim_{n \rightarrow \infty} \frac{(d-1)!}{n^{d-1}} \lambda_R(\mathbf{H}_m^0(\overline{J_i^n}/\overline{J_i^{n+1}})) \right) \\ & = \sum_i (d! \text{vol}(\text{pyr}(J_i)) - j(J_i)) \leq 0. \end{aligned}$$

It follows that

$$\begin{aligned} j(I) &= \lim_{n \rightarrow \infty} \frac{(d-1)!}{n^{d-1}} \lambda_R(\mathbf{H}_m^0(\overline{I^n}/\overline{I^{n+1}})) \\ &\geq \lim_{n \rightarrow \infty} \frac{(d-1)!}{n^{d-1}} \#(\text{cone}_n(I) \cap \mathbb{Z}^d) = d! \text{vol}(\text{pyr}(I)), \end{aligned}$$

where the last equality follows from Remark 2.7.

We subsequently assume that the Newton polyhedron of I has a single bounded facet \mathcal{P} .

Step 2b: Description of a rational polytope containing points contributing to $j(I)$:

Let \mathcal{P} be $\text{bd}(I)$ and let H be the affine $(d-1)$ -plane spanned by \mathcal{P} . Let $\langle x, b \rangle = c$ be a defining equation for H . Recall that each entry of b is positive for a bounded face, so that after rescaling b , we may assume that $c = 1$, with each $b_j > 0$.

We now describe a region $\mathcal{R}_n \subset \text{cone}_n(\mathcal{P})$ such that for any $\alpha \in \mathbb{Z}^d \cap \mathcal{R}_n$, $x^\alpha \in (\overline{I^{n+1}} :_{\overline{I^n}} \mathbf{m}^\infty)$. Note that $x^\alpha \in (\overline{I^{n+1}} :_R x_i^\infty)$ if and only if α is in the image of π_i , the projection in the e_i direction onto $\langle \alpha, b \rangle H$. That is,

$$\Gamma(\overline{I^{n+1}} :_R x_i^\infty) \cap \langle \alpha, b \rangle H = \mathbb{Z}^d \cap \pi_i((n+1)\mathcal{P}).$$

Let $\mathcal{P} = \text{conv}(v_1, \dots, v_t)$. Then

$$\pi_i((n+1)v_j) = (n+1)v_j - \frac{(n+1) - \langle \alpha, b \rangle}{\langle e_i, b \rangle} e_i,$$

so that

$$\pi_i((n+1)\mathcal{P}) = \text{conv} \left((n+1)v_1 - \frac{(n+1) - \langle \alpha, b \rangle}{\langle e_i, b \rangle} e_i, \dots, (n+1)v_t - \frac{(n+1) - \langle \alpha, b \rangle}{\langle e_i, b \rangle} e_i \right)$$

Since each $b_j > 0$, this is well-defined. Now,

$$(\overline{I^{n+1}} :_{\overline{I^n}} \mathbf{m}^\infty) = \overline{I^n} \cap \bigcap_{i=1}^d (\overline{I^{n+1}} :_R x_i^\infty)$$

We define a region

$$\mathcal{R}_n := \text{cone}_n(\mathcal{P}) \cap \bigcap_{i=1}^d ((n+1)\mathcal{P} + \mathbb{R}_{\leq 0} e_i)$$

so that, by the above, $\mathcal{R}_n \cap \mathbb{Z}^d$ consists of all the points α in $\Gamma((\overline{I^{n+1}} :_{\overline{I^n}} \mathbf{m}^\infty)/\overline{I^{n+1}})$.

We remark that for $n \leq s < (n+1)$,

$$\mathcal{R}_n \cap sH \supseteq sH \cap (\mathcal{R}_n \cap nH + \mathbb{R}_{\geq 0}^d) \supseteq \frac{s}{n}(\mathcal{R}_n \cap nH) = \frac{s}{n}(\mathcal{R}_n \cap n\mathcal{P})$$

so that if we set $\tau_n = \frac{1}{n}(\mathcal{R}_n \cap n\mathcal{P})$, then $\text{cone}_n(\tau_n) \subset \mathcal{R}_n$. Note that τ_n has vertices in \mathbb{Q}^d .

We also claim that $\tau_n \subset \tau_{n+1}$. We have $\alpha \in \tau_n$ if and only if

$$\frac{n}{n+1}\alpha + \frac{1}{n+1} \frac{1}{\langle e_i, b \rangle} e_i \in \mathcal{P}$$

for all i . By convexity of \mathcal{P} since $\alpha \in \mathcal{P}$, if $\lambda\alpha + (1-\lambda)\frac{1}{\langle e_i, b \rangle} e_i \in \mathcal{P}$ then also $\lambda'\alpha + (1-\lambda')\frac{1}{\langle e_i, b \rangle} e_i \in \mathcal{P}$ for $0 \leq \lambda \leq \lambda' \leq 1$. In particular,

$$\frac{n+1}{n+2}\alpha + \frac{1}{n+2} \frac{1}{\langle e_i, b \rangle} e_i \in \mathcal{P}$$

for all i , so that $\alpha \in \tau_{n+1}$. It follows by induction that $\tau_n \subset \tau_{n'}$ for $n \leq n'$.

Step 2c: Using $\text{pyr}(\tau_n)$ to give a lower bound:

Consider the distance between a vertex of $n\mathcal{P}$ and the corresponding vertex of $\pi_i((n+1)\mathcal{P})$. We compute this distance as

$$|nv_j - ((n+1)v_j - \frac{1}{\langle e_i, b \rangle} e_i)| = |v_j - \frac{1}{\langle e_i, b \rangle} e_i|,$$

which is bounded above uniformly in n by $L := \max_{i,j} \{|v_j - \frac{1}{\langle e_i, b \rangle} e_i|\}$. Then $n\tau_n = \mathcal{R}_n \cap n\mathcal{P}$ is the intersection of $(d+1)$ many polytopes, each of which is the convex hull of t points v'_1, \dots, v'_t such that $|v_j - v'_j| < L$ for all j . That is, each such polytope is an L -shaking of $n\mathcal{P}$ in the sense of Lemma 2.5. Dividing through by n we see that τ_n is the intersection of $(d+1)$ many polytopes that are all $\frac{L}{n}$ -shakings of the polytope \mathcal{P} .

Given $0 < c < 1$, we may now apply Lemma 2.5 in the affine subspace H . We obtain, for a sufficiently large M , an τ_M such that $\text{vol}(\tau_M) \geq c \text{vol}(\mathcal{P})$ in H , and hence $\text{vol}(\text{pyr}(\tau_M)) \geq c \text{vol}(\text{pyr}(\mathcal{P}))$.

For $n > M$, we have from the previous step that $\tau_n \supseteq \tau_M$, so

$$\text{cone}_n(\tau_M) \subseteq \text{cone}_n(\tau_n) \subseteq \mathcal{R}_n.$$

Thus, if $\alpha \in \mathbb{Z}^d \cap \text{cone}_n(\tau_M)$, then

$$x^\alpha \in (\overline{I^{n+1}} :_{\overline{I^n}} \mathfrak{m}^\infty) / \overline{I^{n+1}} = H_{\mathfrak{m}}^0(\overline{I^n} / \overline{I^{n+1}}).$$

Thus,

$$\begin{aligned} j(I) &= \lim_{n \rightarrow \infty} \frac{(d-1)!}{n^{d-1}} \lambda_R(H_{\mathfrak{m}}^0(\overline{I^n} / \overline{I^{n+1}})) \\ &\geq \lim_{n \rightarrow \infty} \frac{(d-1)!}{n^{d-1}} \#(\mathbb{Z}^d \cap \text{cone}_n(\tau_M)) = d! \text{vol}(\text{pyr}(\tau_M)), \end{aligned}$$

where the last equality follows from Proposition 2.6. Therefore, for all $c < 1$, we have the inequality $j(I) \geq c(d! \text{vol}(\text{pyr}(\mathcal{P})))$, so

$$j(I) \geq d! \text{vol}(\text{pyr}(\mathcal{P})) = d! \text{vol}(\text{pyr}(I)),$$

as required. \square

Remark 3.3. If I is an \mathfrak{m} -primary monomial ideal, the j -multiplicity is equal to the Hilbert-Samuel multiplicity, and $\text{pyr}(I)$ is the complement of the Newton polyhedron in $\mathbb{R}_{\geq 0}^d$. In this way, Theorem 3.2 agrees with Tessier's result for \mathfrak{m} -primary monomial ideals.

4. NORMAL AFFINE SEMIGROUP RINGS

We next record that, with slight modifications, we can use the same proof to establish a similar result for a wider class of rings. In this section, we state this generalization, and describe necessary modifications to the argument.

By an *affine semigroup ring*, we mean a ring $A = k[Q]$ that has a k vector space basis $\{x^q \mid q \in Q\}$, where Q is a subsemigroup of \mathbb{Z}^d (with the operation $+$), and multiplication given by $x^{q_1} x^{q_2} = x^{q_1 + q_2}$. We denote by \mathfrak{m}_A its maximal homogeneous ideal. If A is a normal ring, then there is a cone $\sigma \subseteq \mathbb{R}^d$ with finitely many extremal rays, each of which contains a lattice point (σ is a *rational cone*) and such that $A \cong k[\mathbb{Z}^d \cap \sigma]$, see [11, Chapters 7 and 10]. We assume henceforth that A is presented in this form. Additionally, suppose that σ is *pointed*, i.e., that it contains no nontrivial linear subspace of \mathbb{R}^d , and that $\dim(\sigma) = d$.

Set r_1, \dots, r_s to be *ray generators* for σ , i.e., minimal lattice points along the extremal rays of σ . Let I be a monomial ideal of R minimally generated by $x^{v_1}, x^{v_2}, \dots, x^{v_n}$. In this context, we define

$$\text{conv}(I) := \text{conv}(v_1, \dots, v_n) + \sigma,$$

and, as in section two, $H_i = \{x \in \mathbb{R}^n \mid \langle x, b_i \rangle = c_i\}$, with $b_i \in \mathbb{Q}^d$, $c_i \in \mathbb{Q}$ for $i = 1, \dots, w$ to be the supporting hyperplanes of $\text{conv}(I)$ so that

$$\text{conv}(I) = H_1^+ \cap H_2^+ \cap \dots \cap H_w^+,$$

where $H_i^+ = \{x \in \mathbb{R}^n \mid \langle x, b_i \rangle \geq c_i\}$. We again assume that H_1, \dots, H_w are the hyperplanes corresponding to unbounded facets. We retain the other definitions, e.g., pyr and cone_n , from the preliminary section. Note that since we have chosen an embedding of our semigroup in $\mathbb{Z}^d \subset \mathbb{R}^d$, it makes sense to talk about volume. Our main result from the previous section holds in this context:

Theorem 4.1. *Let $A = k[\mathbb{Z}^d \cap \sigma]_{\mathfrak{m}_A}$, where σ is a d -dimensional pointed rational cone. Let $I \subset A$ be a monomial ideal. Then $j(I) = d! \text{vol}(\text{pyr}(I))$.*

Proof. The proof of Theorem 3.2 applies, after some slight changes:

It is easy to see that the inequalities $\langle r_j, b_i \rangle \geq 0$ hold for all i, j , and $\langle r_j, b_i \rangle > 0$ for $u + 1 \leq i \leq w$ and all j , as in [15, Lemma 1.1]. Apply these inequalities in the proof of Lemma 2.3 mutatis mutandis to obtain the same conclusion. Step 1 in the proof of Theorem 3.2 follows.

To prove the other inequality in this setting, first note that for any monomial ideal J , we have $\alpha \in \Gamma(J :_A \mathfrak{m}_A^\infty)$ if and only if there exist $a_j \in \mathbb{R}_{\geq 0}$ such that $\alpha + a_j r_j \in \Gamma(J)$ for all j . Thus, for a monomial ideal I with a single bounded face \mathcal{P} , we may define the region

$$\mathcal{R}_n := \text{cone}_n(\mathcal{P}) \cap \bigcap_{j=1}^m ((n+1)\mathcal{P} + \mathbb{R}_{\leq 0} r_j)$$

so that $\Gamma(\overline{I^{n+1}} :_{\overline{I^n}} \mathfrak{m}^\infty) / \overline{I^{n+1}} = \mathbb{Z}^d \cap \mathcal{R}_n$. This region has all of the salient properties from the case of the polynomial ring R (i.e., \mathcal{R}_n is a rational polytope such that $\mathcal{R}_n \cap s\mathcal{P} \supseteq \frac{s}{n}(\mathcal{R}_n \cap n\mathcal{P})$ for $n \leq s < (n+1)$), and we employ this to complete the proof of Step 2 as before. \square

Remark 4.2. The j -multiplicity of an ideal I is greater than zero if and only if I has maximal analytic spread ([12, Lemma 3.1]). Then it follows from Theorems 3.2 and 4.1 that a monomial ideal of a normal affine semigroup ring has maximal analytic

spread if and only if $\text{conv}(I)$ has some bounded facet of dimension $d - 1$. This is a fact that in the case of polynomial rings is a direct consequence of [2, Theorem 2.3] or [15, Corollary 4.10].

5. THE ε -MULTIPLICITY AS A VOLUME

In this section we follow the notation from the third section, i.e., $R = k[x_1, \dots, x_d]_{\mathfrak{m}}$. The ε -multiplicity was defined by Ulrich and Validashti [18] in 2011 as a generalization of the Buchsbaum-Rim multiplicity for submodules of free modules with arbitrary colength. In its simpler form for ideals, the ε -multiplicity is defined by

$$\varepsilon(I) = \limsup_{n \rightarrow \infty} \frac{d!}{n^d} \lambda_R(H_{\mathfrak{m}}^0(R/I^n)),$$

where the limit of the sequence has been shown to exist in wide generality, see [5]. For monomial ideals, the limit is known to exist and is a rational number as shown in [9, Corollary 2.5]. Nevertheless, unlike the j -multiplicity, there are examples of ideals for which the ε -multiplicity is not an integer; see [6, Example 2.4]. In this section, we will give a combinatorial proof of the existence and rationality of the limit in the monomial case, identifying $\varepsilon(I)$ with the normalized volume of a region with rational vertices.

Let $\text{out}(I)$ be the region $(H_1^+ \cap \dots \cap H_u^+) \cap (H_{u+1}^- \cup \dots \cup H_w^-)$ if the ideal I has maximal analytic spread, and empty otherwise. Notice that by Lemma 2.3, $\text{out}(I)$ is bounded.

Theorem 5.1. *Let $I \subset R$ be a monomial ideal. Then the limit in the definition of $\varepsilon(I)$ exists, and $\varepsilon(I) = d! \text{vol}(\text{out}(I))$, which is a rational number.*

Proof. Since the functor $H_{\mathfrak{m}}^0(-)$ is sub-additive on short exact sequences, we have

$$\lambda_R(H_{\mathfrak{m}}^0(R/I^n)) \leq \sum_{i=0}^{n-1} \lambda_R(H_{\mathfrak{m}}^0(I^i/I^{i+1})).$$

The right hand side is the sum transform of the function that defines the j -multiplicity, and hence for $n \gg 0$ it is equal to a polynomial of degree d and leading coefficient $\frac{j(I)}{d!}$, see [12]. It follows that $\varepsilon(I) \leq j(I)$, so we can assume that I has maximal analytic spread.

Step a: Existence of the limit for the filtration $\{\overline{I^n}\}_{n \in \mathbb{N}}$:

By Lemma 2.3,

$$\Gamma((\overline{I^n} :_R \mathfrak{m}^\infty) / \overline{I^n}) = \mathbb{Z}^d \cap (\text{out}(I^n) \setminus \text{bd}(I^n)).$$

Let $\mathcal{Q} = \text{conv}(\text{vert}(I))$ and $\mathcal{Q}' = \text{conv}(\mathcal{Q}, \text{out}(I))$; it is easy to check that there is an equality $\mathcal{Q}' \setminus \mathcal{Q} = \text{out}(I) \setminus \text{bd}(I)$. By [15, Lemma 3.3], the hyperplanes $\{nH_i\}$, for $1 \leq i \leq w$, are the supporting hyperplanes of $\text{conv}(I^n)$ for each $n \geq 1$. Then we also have $n\mathcal{Q}' \setminus n\mathcal{Q} = \text{out}(I^n) \setminus \text{bd}(I^n)$.

Hence,

$$\begin{aligned} \lambda_R(H_{\mathfrak{m}}^0(R/\overline{I^n})) &= \#(\mathbb{Z}^d \cap (\text{out}(I^n) \setminus \text{bd}(I^n))) \\ &= \#(\mathbb{Z}^d \cap (n\mathcal{Q}' \setminus n\mathcal{Q})) \\ &= E_{\mathcal{Q}'}(n) - E_{\mathcal{Q}}(n), \end{aligned}$$

where the latter is the difference of two Ehrhart quasi-polynomials of the form

$$(\operatorname{vol}(\mathcal{Q}') - \operatorname{vol}(\mathcal{Q}))n^d + O(n^{d-1}) = \operatorname{vol}(\operatorname{out}(I))n^d + O(n^{d-1})$$

(see proof of Lemma 2.6), and the result follows.

Step b: Existence of the original limit:

By [19, Theorem 7.29], $\overline{I^{n+1}} = \overline{I} \overline{I^n}$ for $n \geq d$. Then we have the following exact sequences for $n \geq d$:

$$0 \rightarrow \overline{I^n}/I^n \rightarrow R/I^n \rightarrow R/\overline{I^n} \rightarrow 0,$$

$$0 \rightarrow I^n/\overline{I^{n+d}} \rightarrow R/\overline{I^{n+d}} \rightarrow R/I^n \rightarrow 0,$$

and the following inequalities

$$\begin{aligned} (1) \quad \lambda_R(\mathrm{H}_{\mathfrak{m}}^0(R/\overline{I^{n+d}})) &\leq \lambda_R(\mathrm{H}_{\mathfrak{m}}^0(R/I^n)) + \lambda_R(\mathrm{H}_{\mathfrak{m}}^0(I^n/\overline{I^{n+d}})) \\ (2) \quad &\leq \lambda_R(\mathrm{H}_{\mathfrak{m}}^0(R/\overline{I^n})) + \lambda_R(\mathrm{H}_{\mathfrak{m}}^0(\overline{I^n}/I^n)) + \lambda_R(\mathrm{H}_{\mathfrak{m}}^0(I^n/\overline{I^{n+d}})). \end{aligned}$$

Now, $\lambda_R(\mathrm{H}_{\mathfrak{m}}^0(I^n/\overline{I^{n+d}})) \leq \lambda_R(\mathrm{H}_{\mathfrak{m}}^0(\overline{I^n}/\overline{I^{n+d}})) \leq \sum_{i=0}^{d-1} \lambda_R(\mathrm{H}_{\mathfrak{m}}^0(\overline{I^{n+i}}/\overline{I^{n+i+1}}))$, so

$$\limsup_{n \rightarrow \infty} \frac{d!}{n^d} \lambda_R(\mathrm{H}_{\mathfrak{m}}^0(I^n/\overline{I^{n+d}})) \leq \sum_{i=0}^{d-1} \limsup_{n \rightarrow \infty} \frac{d!}{n^d} \lambda_R(\mathrm{H}_{\mathfrak{m}}^0(\overline{I^{n+i}}/\overline{I^{n+i+1}})) = 0,$$

where the last equality holds by Proposition 3.1.

Therefore, $\lim_{n \rightarrow \infty} \frac{d!}{n^d} \lambda_R(\mathrm{H}_{\mathfrak{m}}^0(I^n/\overline{I^{n+d}})) = 0$.

Similarly, for $n \geq d$,

$$\lambda_R(\mathrm{H}_{\mathfrak{m}}^0(\overline{I^n}/I^n)) \leq \lambda_R(\mathrm{H}_{\mathfrak{m}}^0(I^{n-d}/I^n)) \leq \sum_{i=0}^{d-1} \lambda_R(\mathrm{H}_{\mathfrak{m}}^0(I^{n-d+i}/I^{n-d+i+1})),$$

and then $\lim_{n \rightarrow \infty} \frac{d!}{n^d} \lambda_R(\mathrm{H}_{\mathfrak{m}}^0(\overline{I^n}/I^n)) = 0$.

Using these two limits in (1) and (2), we obtain

$$\begin{aligned} d! \operatorname{vol}(\operatorname{out}(I)) &= \liminf_{n \rightarrow \infty} \frac{d!}{n^d} \lambda_R(\mathrm{H}_{\mathfrak{m}}^0(R/\overline{I^{n+d}})) \\ &\leq \liminf_{n \rightarrow \infty} \frac{d!}{n^d} \lambda_R(\mathrm{H}_{\mathfrak{m}}^0(R/I^n)) \\ &\leq \limsup_{n \rightarrow \infty} \frac{d!}{n^d} \lambda_R(\mathrm{H}_{\mathfrak{m}}^0(R/I^n)) \\ &\leq \limsup_{n \rightarrow \infty} \frac{d!}{n^d} \lambda_R(\mathrm{H}_{\mathfrak{m}}^0(R/\overline{I^n})) = d! \operatorname{vol}(\operatorname{out}(I)) \end{aligned}$$

which finishes the proof. \square

6. EXAMPLES

Example 6.1. Let $I = (x^4, x^2y, xy^2)$. We compute the j -multiplicity as two times the area of the green region in Figure 2, obtaining $j(I) = 7$. For the ε -multiplicity, we take two times the area of the portion of the green region that lies above the dotted line, obtaining $\varepsilon(I) = 5$.

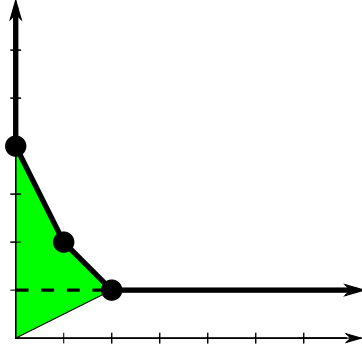


FIGURE 2.

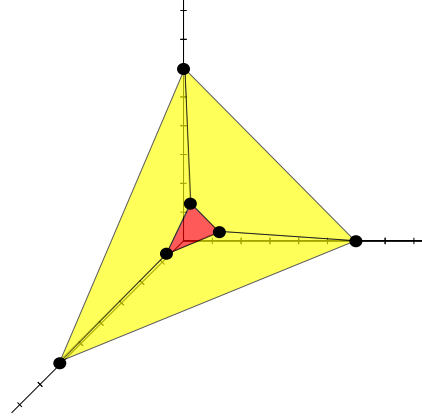


FIGURE 3.

Example 6.2. Let $I = (x^6, y^6, z^6, x^2yz, xy^2z, xyz^2)$. This monomial ideal is \mathfrak{m} -primary, so we can compute its Hilbert-Samuel multiplicity as the volume underneath the bounded faces of its Newton polyhedron, which is depicted in Figure 3. We decompose this region as three regions under the yellow faces corresponding to $I_1 = (x^6, y^6, x^2yz, xy^2z)$, $I_2 = (y^6, z^6, xy^2z, xyz^2)$, $I_3 = (x^6, z^6, x^2yz, xyz^2)$, and one region under the red face corresponding to $I_4 = (x^2yz, xy^2z, xyz^2)$. We compute $j(I_1) = j(I_2) = j(I_3) = 42$, $j(I_4) = 4$, so that $e(I) = j(I) = 130$.

Example 6.3. [6, Example 2.4] Let $I = (xy, yz, zx)$. The region $\text{out}(I)$ is the tetrahedron with vertices $\{(1, 0, 1), (1, 1, 0), (0, 1, 1), (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})\}$, and its volume is $\frac{1}{12}$. Thus, $\varepsilon(I) = \frac{1}{2}$.

Example 6.4. For a graph G on the vertex set $\{1, \dots, d\}$, the *edge ideal* of G is the monomial ideal

$$I_G := (x_i x_j \mid \{i, j\} \text{ is an edge of } G) \subset k[x_1, \dots, x_d].$$

The region $\text{bd}(I_G)$ is known in the literature as the *edge polytope* of G , see [13]. By Theorem 3.2, the j -multiplicity $j(I_G)$ is $(d-1)! \cdot h$ times the volume of the edge polytope of G , where h is the distance from the origin to the plane $\sum x_i = 1$. As a particular example, let C_d be the cycle on d vertices. Then,

$$j(I_{C_d}) = \begin{cases} 0 & \text{if } d \text{ is even} \\ 2 & \text{if } d \text{ is odd.} \end{cases}$$

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